Rational homotopy theory

§1

Idea: Rational homotopy theory ⟷ Homotopy theory mod torsion
↳ Special case of homotopy theory mod Serre class.

In classical homotopy theory, we don't know all π_* for any non-contractible simply connected finite CW-complex. But, we know

$$\pi_n(X) \cong \mathbb{Z}^r \oplus T \quad \text{for } X \text{ 1-conn. finite CW-complex.}$$

$\rightsquigarrow \pi_n(X) \otimes \mathbb{Q} \cong \mathbb{Q}^r$ rational vector space.

Def / Thm: [Whitehead mod T] $Y, X \in \mathrm{Top}_2$

$f : X \to Y$ is a rational homotopy equivalence, if

$$\pi_* f \otimes \mathbb{Q} : \pi_* X \otimes \mathbb{Q} \xrightarrow{\cong} \pi_* Y \otimes \mathbb{Q} \quad \text{or equiv.}$$

$$H_*(f, \mathbb{Q}) : H_*(X ; \mathbb{Q}) \xrightarrow{\cong} H_*(Y ; \mathbb{Q}).$$

Def: A simply connected space is called rational, if $\pi_k X$ are uniquely divisible $\forall k$.

Thm: Every $X \in \mathrm{Top}_2$ admits a rationalisation, i.e. $X_\mathbb{Q}$ s.t.

$$\pi_k(X) \otimes \mathbb{Q} \cong \pi_k(X_\mathbb{Q}) \quad \forall k.$$

In diagrams:

$$\begin{tikzcd}
\mathrm{Top}_2^{\mathbb{Q}} \arrow[r, hook] \arrow[d] & \mathrm{Top}_2 \arrow[d] \\
\mathrm{Top}_2^{\mathbb{Q}}[W_\mathbb{Q}^{-1}] \arrow[r, "\cong"] & \mathrm{Top}_2[W_\mathbb{Q}^{-1}]
\end{tikzcd}$$

§2 Quillen's paper

Def: Define a graded Lie algebra

$$\Pi_q X := \pi_q \Omega X \otimes \mathbb{Q} \cong \pi_{q+1} X \otimes \mathbb{Q}$$

with bracket the Whitehead product

Thm: [Quillen]

There is a chain of Quillen equivalences

$$\mathrm{Top}_2 \xrightarrow{\lambda} \mathrm{dg}\mathcal{L}\mathrm{ie} \; (\xrightarrow{C_*} \mathrm{dg}\mathcal{C}\mathcal{C} \;)$$

↳ Chevalley - Eilenberg

↦ homology, diff of deg -1

Moreover $\Pi_* X \cong H \tilde{\lambda} X$ as functors $\mathrm{Ho}_\mathbb{Q} \mathrm{Top}_2 \longrightarrow \mathrm{gr}\,\mathcal{L}\mathrm{ie}$

$\tilde{\lambda} : \mathrm{Ho}_\mathbb{Q} \mathrm{Top}_2 \longrightarrow \mathrm{Ho}\, \mathrm{dg}\mathcal{L}\mathrm{ie}_{\geq 1}$ induced equivalence on htpy cats

Whitehead bracket: bilinear pairing

$$[\,,\,] : \pi_p X \times \pi_q X \to \pi_{p+q-1} X$$

$$\begin{tikzcd}
\alpha\,,\,\beta \arrow[r, mapsto] & S^{p+q-1} \arrow[r, "\omega"] & S^p \vee S^q \arrow[d, "\alpha \vee \beta"] \\
& & X
\end{tikzcd}$$

$$\begin{tikzcd}
D^p \times D^q \arrow[r, "\partial \mapsto *"] & S^p \times S^q \arrow[d] \\
S^{p+q-1} \cong \partial(D^p \times D^q) \arrow[u, hook] \arrow[r, dashed] \arrow[d, "\wr"] & S^p \vee S^q \\
\text{image } \partial(D^p \times D^q) &
\end{tikzcd}$$

$\cap$

$S^p \vee S^q \subset S^p \times S^q$

$$\begin{tikzcd} \mathcal{T}op_2^{*/} \arrow[r, shift right, "E_2\mathcal{S}ing"'] & s\mathcal{S}et_2 \arrow[l, shift right, "|-|"'] \arrow[r, shift left, "G"] & s\mathcal{G}rp_1 \arrow[l, shift left, "\overline{W}"] \arrow[r, shift left, "\hat{\mathbb{Q}}"] & s\mathcal{CHA}_1 \arrow[l, shift left, "\mathcal{G}"] \arrow[r, shift right, "\mathcal{P}"'] & s\mathcal{LA}_1 \arrow[l, shift right, "\hat{\mathcal{U}}"'] \arrow[r, shift right, "N"'] & dg\mathcal{LA} \arrow[l, shift right, "N^*"'] \arrow[r, shift right, "\mathcal{C}"'] & dg\,\mathcal{C}o\mathcal{A}lg \arrow[l, shift right, "\mathcal{L}"'] \end{tikzcd}$$

$|S| \longleftarrow\!\shortmid\; S$

$X \mapsto \mathcal{S}ing X \mapsto E_2\mathcal{S}ing X$

$$\begin{tikzcd} s\mathcal{S}et^{*/} \arrow[r, shift left, hook] \arrow[r, shift right, "E_2"'] & s\mathcal{S}et_2 \end{tikzcd}$$

E_2 "
1-simplices
are constant at *
does not! change
htpy type.

$G \longmapsto \hat{\mathbb{Q}}G = \lim {}^{\mathbb{Q}G}/_{\mathbb{Q}G^n}$

the group algebra completed.

$\mathcal{G}A \longleftarrow\!\shortmid\; A$

$\mathcal{G}A = \{x \in 1 \oplus \bar{A} \mid \Delta x = x \otimes x\}$

the group-like elements.

$N^*\mathfrak{g} \longleftarrow\!\shortmid\; \mathfrak{g}$

adjoint of N

$\mathfrak{g} \longmapsto N\mathfrak{g}$

$(N_q\mathfrak{g} = \bigcap_{0<j\leq q} \ker d_j ,\ \partial = d_0)$

(Moore complex)
Bracket from
Eilenberg-Zilber
formula.

$S \longmapsto GS$

$G_nS = \frac{FS_{n+1}}{FS_n}$

think of Ω.

$\hat{\mathcal{U}}\mathfrak{g} = \lim {}^{\mathcal{U}\mathfrak{g}}/_{\mathcal{I}\mathfrak{g}^n} \longleftarrow\!\shortmid\; \mathfrak{g}$

completion of the
universal enveloping
algebra

$\overline{W}G \longleftarrow\!\shortmid\; G$

$WG_n = \prod_{i=0}^{n} G_i$

$WG \xrightarrow[\simeq]{\text{Fib.}} \overline{W}G = {}_G\backslash^{WG}$

think of B

$A \longmapsto \mathcal{P}A$

$\mathcal{P}A = \{x \in \bar{A} \mid \Delta x = x \otimes 1 + 1 \otimes x\}$

the primitives.

Equivalence:

Milner Kan Curtis' convergence theorem and Quillen Dold-Puppe

Cor: If L is a reduced, graded Lie alg, then $L \cong \pi X$ for some $X \in \mathcal{T}op_2$.

Pf: $(L, 0) \in dg\mathcal{L}ie$. By thm, $\exists X \in \mathcal{T}op_2 : \tilde{\lambda} X = L$

$\Rightarrow \pi X \cong H \tilde{\lambda} X \cong HL = L$.

commutative dg alg
$\uparrow$ Edgel
via Sullivan's
Lie models,
Quillen to Koszul
dual to Sullivan
[Rational htpy th
Hess]

Examples of Lie models (Ch. 24 : Rational homotopy theory)

1) $\left(\mathbb{L}(v) = \begin{cases} \mathbb{Q}v & \text{, if } \deg v = 2n \\ \mathbb{Q}v \oplus \mathbb{Q}[v,v] & \text{, if } \deg v = 2n+1 \end{cases} \subset T(v), 0\right)$

is a Lie model for S^{n+1}.

2) The ~~free product~~ direct sum of Lie models is a Lie model for the ~~wedge~~ product.

$\{L_\alpha\} \subset g\mathcal{L}ie \rightsquigarrow \coprod_\alpha L_\alpha = L$ s.t.
$V = \bigoplus L_\alpha$, $i_\alpha : L_\alpha \hookrightarrow V$, $I = i_\alpha[x,y] - [i_\alpha x, i_\alpha$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cap$
$\rightsquigarrow L = \mathbb{L}_{V/I}$. $\qquad\qquad\qquad\qquad \mathbb{L}_V$

3) X CW cplx with no 1-cells. $\rightsquigarrow (\mathbb{L}_V, d)$ Lie model s.t.

- $(\mathbb{L}_{V<n}, d)$ is a Lie model for X_n. $\longrightarrow$ Compare last page.
- A basis $\{v_\alpha\} \subset V_n$ corresponds to the D_α^{n+1} - cells of X.
- $s[dv_\alpha] \in sH(\mathbb{L}_{V<n})$ corresponds to $[f_\alpha]$ - attaching maps.

under $\tau_c : sH(\mathbb{L}_{V<n}) \xrightarrow{\cong} \pi_*(\Omega X_n) \otimes \mathbb{Q}$.

Construction of Quillen adjunctions.

Model structures:

$sSet_2$ - 2-reduced simplicial sets, ie. $S_0, S_1 = cst_*$

$\mathcal{W} = \{ f : S \to T \mid |f| \in \mathcal{W}_\mathbb{Q} \}$

$\mathcal{C}of$ = levelwise injectives

$\mathcal{F}ib$ = RLP against $\mathcal{C}of \cap \mathcal{W}$.

$sGrp_1$ - 1-reduced simplicial groups, ie. $G_0 = \{e\}$

$\mathcal{W} = \{ f : G \to H \mid \pi_* f \otimes \mathbb{Q} \text{ iso} \}$, $\quad NG_n = \bigcap_{i=1}^{n} \ker d_i$, $\partial = d_0|_{(NG)_n}$

$\mathcal{C}of$ = Retracts of free simplicial grps $\qquad \pi_n G := H_{n-1}(NG)$

$\mathcal{F}ib$ = RLP against $\mathcal{C}of \cap \mathcal{W}$

Remark: G maps ~~$\mathcal{F}ib$~~ $\mathcal{C}of$ to ~~$\mathcal{F}ib$~~ $\mathcal{C}of$:

$\{ f_n : S_n \rightarrowtail T_n \} \longmapsto Gf_n : FS_{n+1}/FS_n \longrightarrow FT_{n+1}/T_n$

$\qquad\qquad \cap$
$\qquad\quad$ ~~$\mathcal{F}ib$~~
$\qquad\quad \mathcal{C}of$

$$\begin{tikzcd} FS_{n+1}/FS_n \arrow[d, rightarrowtail] & FS_{n+1} \arrow[l, hook'] \arrow[r] \arrow[d, rightarrowtail] & FS_{n+1}/FS_n \arrow[d, rightarrowtail] \\ FT_{n+1}/ & FT_{n+1} \arrow[l, hook'] \arrow[r] & FT_{n+1}/FT \end{tikzcd}$$

Interlude: complete Hopf algebras

Def: A complete augmented algebra A is an algebra with

- augmentation $\varepsilon : A \to k$
- filtration $A = F_0 A \supset F_1 A \supset \dots$ st. $F_p A \cdot F_r A \subset F_{p+r} A$

st.

i) $F_1 A = \bar{A} = \ker \varepsilon$

ii) $\mathrm{gr}\, A$ is generated by $\mathrm{gr}_1 A = {F_1 A}/{F_2 A} = {\bar{A}}/{F_2 A}$

iii) $A = \lim {A}/{F_n A}$ (complete wrt filtration)

A complete Hopf algebra is a complete augmented algebra with diagonal

$$\Delta : A \to A \hat{\otimes} A,$$

completion wrt. filtration
$F_n(V \otimes W) = \bigoplus_{i+j=n} F_i V \otimes F_j W.$

a map of caa that is cocommutative and coassociative.

Remark: The existence of antipode is automatic in this setting, since $\bar{A}$ is a conilpotent Bialg, it has antipode that extends by completeness

Examples:

- $k[\![x]\!]$

  $\varepsilon : k[\![x]\!] \to k$ ✓
  $F_1 A = k[\![x]\!]/k$, $F_2 A = k[\![x]\!]/(k, x)$, ...
  i) ✓ ii) gen. by x ✓ iii) $A = \lim {A}/{F_n A}$ ✓
  $\Delta : x \mapsto x \otimes 1 + 1 \otimes x$

- A augm. alg / Hopf alg. $\Rightarrow \hat{A} := {}^{A}$
- eg. $\widehat{\mathcal{U}\mathfrak{g}}$ for $\mathfrak{g}$ Lie algebra.

Model structures continued:

$s\mathcal{CHA}_1$

$\mathcal{W} = \{ f \mid \mathcal{P} f \text{ is a weak equivalence} \}$

$\mathcal{F}ib = \{ f \mid \mathcal{P} f \text{ is a fibration} \}$

$\mathcal{C}of$ = Retracts of free maps.

Actually, same as downstairs, but $k[\![x]\!]$ is a proj. generator that is co-Lie alg object.

$\Rightarrow$ easy that $\mathcal{P}$ preserves fibs and acyclic fibs

$\Rightarrow$ Quillen adjunctions.

$s\mathcal{LA}_1$

$\mathcal{W} = \{ f \mid f_* : \underline{\mathrm{Hom}}(P, X) \to \underline{\mathrm{Hom}}(P, Y) \text{ is a whe for } \forall P \text{ projective} \}$

$\mathcal{F}ib = \{ f \mid f_* : \underline{\mathrm{Hom}}(P, X) \to \underline{\mathrm{Hom}}(P, Y) \text{ is a fibration } \forall P \text{ proj} \}$

$\mathcal{C}of$ = Retracts of free maps.

$\mathrm{dg}\mathcal{L}\mathcal{A}_1$

$\mathcal{W}$ = Quasi-isomorphism

$\mathcal{F}ib$ = surjective in deg > 1

$\mathcal{C}of$ = LLP against $\mathcal{W} \cap \mathcal{F}ib$ $\Leftrightarrow$ retract of a free map

$\overset{!}{\Leftarrow}$ free maps

↓ sub-graded Lie algs of free reduced Lie algs are free

$S(q) = \mathbb{Q}\sigma_q$, $|\sigma_q| = q$, $D(q) = \mathbb{Q}\sigma_{q-1} \oplus \mathbb{Q}\tau_q$ $|\tau_q| = q$

$f: \mathfrak{g} \to \mathfrak{m}$ free if $\mathfrak{m} \cong \mathfrak{g} \oplus \mathbb{L}_V$ and $f \cong$ incl. $d\tau_q = \sigma_{q-1}$

$d\sigma_{q-1} = 0$.

n-skel of f $\mathfrak{m}^n := \langle f(\mathfrak{m}), v \in V \mid |v| \leq n \rangle \Rightarrow \mathfrak{m}^n = \mathfrak{m}^{n-1} \amalg_{\mathbb{L}_{S(n-1)}} \mathbb{L}_{D(n)}$

↗ for $n > r$, $\mathbb{L}_{S(n-1)} \to \mathbb{L}_{D(n)}$ and $0 \to \mathbb{L}_{S(r)}$ are cofibs

$\Rightarrow$ any free map is. incl. $\Rightarrow f = pi$ for any f

$\overset{\downarrow}{p}$ acyclic, $\overset{\searrow}{i}$ free (*

$f \in \mathcal{C}of \Rightarrow f$ retract of i

Top_2 "model structure" Top_2 is not closed under limits.

$\mathcal{W}$ = rational homotopy equivalences

$\mathcal{C}of$ = cellular complexes / CW-complexes

$\mathcal{F}ib$ = Serre fibrations with rational fibres.

Remark:

Top_2 is not closed under limits, but we have a work-around making $|-| \dashv E_2\mathrm{Sing}$ a "Quillen equivalence" - preserving the homotopy theory:

The cofibre sequences have no problems, as we may take cofibrant replacements (with non-deg. base pts)

The fibre sequences work by taking a replacement of the fibre $F \simeq |E_r F|$ that is weakly equivalent.

What can we use this for?

- Algebraic version of topological invariant.
- Transfer some spectral sequences, as Quillen does in his paper.
- DAG $\underline{\mathrm{XIII}}$: How does this fit into our deformation theory?

The equivalence $\mathrm{Lie}_k \longrightarrow \mathrm{Moduli}_k$ (DAG X)

↙ ∞-cat of diff graded Lie algs / k

↓ ∞-cat of formal moduli problems / k

restricts to an equivalence

$$\mathrm{Lie}_k^{\geq 1} \longrightarrow \mathrm{Moduli}_k^{\leq 2}$$

(*) elaborated.

Put $S(q-1) := \mathbb{Q}\sigma_{q-1}$, $|\sigma_{q-1}| = q-1$, $d\sigma_{q-1} = 0$

$D(q) := \mathbb{Q}\sigma_{q-1} \oplus \mathbb{Q}\tau_q$, $|\tau_q| = q$, $d\tau_q = \sigma_{q-1}$

Let $f : \mathfrak{m} \to \mathfrak{n}$ be a free map of dglas, i.e.

$\mathfrak{n} \cong f(\mathfrak{m}) \oplus \mathbb{L}V$ s.t. $f \cong$ inclusion.

The n-skeleton is the graded sub Lie alg.

$\mathfrak{n}^n := \langle f(\mathfrak{m}), V_{\leq n} \rangle$, hence $\mathfrak{m}^n$ is obtained from $\mathfrak{n}^{n-1}$ by

cofib ← for $n>1$

$$\begin{tikzcd} \amalg \mathbb{L}S(n-1) \arrow[r] \arrow[d] & \mathfrak{m}^{n-1} \arrow[d] \\ \amalg \mathbb{L}D(n) \arrow[r] & \mathfrak{n}^n \end{tikzcd}$$

, $0 \to \mathbb{L}S$ cofib

$\Rightarrow$ free map is a cofib.

Mimick the procedure to kill homotopy groups

$\Rightarrow f = \underset{\text{trivial fib}}{p} \circ \underset{\text{free map}}{i}$ for any $f \in \text{Mor}\, dg\mathcal{L}a_{\geq 1}$

And if f is already a cofib, then f has LLP against $p \Rightarrow f$ is a retract of a free map.